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# Normal Hopf subalgebras of semisimple Drinfeld doubles

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ABSTRACT

In this paper we study normal Hopf subalgebras of a semisimple Drinfeld double. This is realized by considering an analogue of Goursat's lemma concerning fusion subcategories of Deligne products of two fusion categories. As an application we show that the Drinfeld double of any abelian extension is also an abelian extension.



## 0. Introduction and the main results

Let $A$ be a semisimple Hopf algebra and $D(A)$ be its Drinfeld double. It is well known that the category $\mathrm{Rep}(D(A))$ of finite dimensional representations of $D(A)$ is a modular tensor category which is braided equivalent to the Drinfeld center of the fusion category $\mathrm{Rep}(A)$ (see [1] or [12]). From this point of view Drinfeld doubles form a special class of quasitriangular Hopf algebras that play a very important role in the classification of semisimple Hopf algebras. Another important notion in the classification of semisimple Hopf algebras is that of a normal Hopf subalgebra of a Hopf algebra. For example, the class of semisolvable Hopf algebras introduced in the paper [19] is constructed starting from a tower of normal Hopf subalgebras.

In this paper we study a class of Hopf subalgebras of $D(A)$ that have normal intersections with both $A$ and $A^*$. They are parametrized by two normal Hopf subalgebras of $A$ and some intermediate group data, see Theorem 3.4. Clearly all normal Hopf subalgebras of $D(A)$ belong to this class. As a consequence we show that if $A$ is simple as a Hopf algebra then all possible normal Hopf subalgebras of $D(A)$ are central group subalgebras of $D(A)$.

In order to show that the Drinfeld double of an abelian extension is a group theoretical Hopf algebra in [20, Theorem 1.3] it is shown that this Drinfeld double is equivalent to an $R$-twist of the (twisted) Drinfeld double of a finite group. We show in Theorem 5.4 that the Drinfeld double of any abelian extension is also an abelian extension. As another application we obtain a description of all minimal normal Hopf subalgebras of a semisimple Drinfeld double $D(A)$.

In order to obtain the classification from Theorem 3.4 we make use of the theory of fusion categories, in particular we apply a highly nontrivial quantum analogue of Goursat's lemma for groups. This quantum analog result first appeared in the unpublished manuscript [8] and it is presented here in Theorem 2.1.

Using this result, first a categorical description for all Hopf subalgebras of $D(A)$ is presented. This enables us to construct a new class of Hopf subalgebras of $D(A)$ in Proposition 3.3. This class is parameterized by two normal Hopf subalgebras $L_1, L_2$ of $A$ and a finite group $G$ with some additional properties. The corresponding Hopf subalgebra of $D(A)$ is denoted by $B(L_1, L_2, G, \psi_1, \psi_2)$. It is shown in Theorem 3.4 that any Hopf subalgebra of $D(A)$ with normal intersections with $A$ and $A^*$ is of the type $B(L_1, L_2, G, \psi_1, \psi_2)$ mentioned above.

Necessary and sufficient conditions for $B(L_1, L_2, 1, 1, 1)$ to be a normal Hopf subalgebra of $D(A)$ are given in Theorem 4.5. Moreover, with these conditions satisfied it is shown that the quotient Hopf algebra $D(A)//B(L_1, L_2, 1, 1, 1)$ is a bicrossed product of the Hopf algebras $L_1^{*\mathrm{cop}}$ and $A//L_2$. Along the way we also obtain some other new results concerning Hopf

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subalgebras of a semisimple Drinfeld double. For example in Theorem 3.1 it is shown that any Hopf subalgebra of $D(A)$ having trivial intersections with $A$ and $A^*$ is a group algebra.

This paper is organized as follows. The first section recalls few basic results on fusion categories and semisimple Hopf algebras that are needed through the rest of the paper. Section 2 presents the quantum analogue of Goursat's lemma that appeared first in [8]. In the next section the Hopf subalgebras $B(L_1, L_2, G, \psi_1, \psi_2)$ of $D(A)$ are constructed. Section 4 shows that any Hopf subalgebra of $D(A)$ with normal Hopf subalgebra intersections with $A$ and $A^*$ is of the form $B(L_1, L_2, G, \psi_1, \psi_2)$. In the last section some examples and applications of the above results are presented. For any abelian extension $A$ the Appendix describes the group structures from Theorem 5.4 that realizes $D(A)$ as an abelian extension.

We work over an algebraic closed field $k$ of characteristic zero. We use a short version $\Delta(x) = x_1 \otimes x_2$ of Sweedler's notation for comultiplication. All the other Hopf algebra notations of this paper are similar to those used in [18].

## 1. Normal fusion subcategories

In this section we recall few basic facts on fusion categories and normal fusion subcategories from [10] and [8] that are needed through the rest of the paper.

### 1.1. General conventions on fusion categories

As usually, by a fusion category we mean a $k$-linear semisimple rigid tensor category $\mathcal{C}$ with finitely many isomorphism classes of simple objects, finite dimensional spaces of morphisms, and such that the unit object of $\mathcal{C}$ is simple. We refer the reader to [9] for a general theory of such categories.

Let $\mathcal{C}$ be a fusion category and denote by $\mathcal{O}(\mathcal{C})$ its set of simple objects considered up to isomorphism. Recall that the Grothendieck ring $K_0(\mathcal{C})$ of $\mathcal{C}$ is the free $\mathbb{Z}$-module generated by the isomorphism classes of simple objects of $\mathcal{C}$ with the multiplication induced by the tensor product in $\mathcal{C}$. The Grothendieck ring $K_0(\mathcal{C})$ is a based unital ring (see for example [10] for definition of based rings). The isomorphism classes $[X]$ of simple objects of $\mathcal{C}$ form a $\mathbb{Z}$-basis for $K_0(\mathcal{C})$.

Let $L_{[X]}$ be the linear operator given by left multiplication by $[X]$ in the based ring $K_0(\mathcal{C})$. Then the Frobenius–Perron dimension of an object $X \in \mathcal{C}$ is defined as the largest positive eigenvalue (Frobenius–Perron eigenvalue) of the matrix associated to $L_{[X]}$ with respect to the linear basis given by $\mathcal{O}(\mathcal{C})$ of $K_0(\mathcal{C})$. This eigenvalue is usually denoted by $\mathrm{FPdim}(X)$.

A fusion subcategory $\mathcal{D}$ of a fusion category $\mathcal{C}$ is a full abelian replete subcategory $\mathcal{D}$ of $\mathcal{C}$ which is closed under the tensor product.

### 1.2. Pointed fusion categories

A fusion category $\mathcal{C}$ is called pointed if any simple object of $\mathcal{C}$ is invertible. It is well known that pointed fusion categories are of the form $\mathrm{Vec}_G^{\omega}$ for a finite group $G$ and some three cocycle $\omega \in H^3(G, k^*)$. Moreover the category of representations $\mathrm{Rep}(H)$ of a semisimple Hopf algebra is pointed if and only if $H$ is commutative, i.e $H = kG^*$ as the Hopf algebra. In this case one has $\mathrm{Rep}(H) = \mathrm{Vec}_G$, i.e. $\omega = 1$.

### 1.3. Universal gradings of fusion categories

Let $\mathcal{C}$ be a fusion category. A grading of a fusion category $\mathcal{C}$ by a group $G$ (or a $G$-grading) is a map $\deg : \mathcal{O}(\mathcal{C}) \to G$ with the following property: for any simple objects $X, Y, Z \in \mathcal{C}$ such that $X \otimes Y$ contains $Z$ one has $\deg(Z) = \deg(X)\deg(Y)$ . This corresponds to a decomposition $\mathcal{C} = \oplus_{g\in G}\mathcal{C}_g$, where $\mathcal{C}_g \subset \mathcal{C}$ is the full additive subcategory generated by the simple objects of $\mathcal{C}$ with degree $g$. The abelian subcategory $\mathcal{C}_1$ is a fusion subcategory of $\mathcal{C}$, called the trivial component of the grading.

Recall that a $G$-grading is said to be trivial if $\mathcal{C}_1 = \mathcal{C}$. It is also said to be faithful if the map $\deg : \mathcal{O}(\mathcal{C}) \to G$ is surjective. For any fusion category $\mathcal{C}$, as explained in [10, Section 3.2], there is a notion of universal grading whose grading group is called the universal grading group of $\mathcal{C}$ and it is denoted by $U_{\mathcal{C}}$. Then the trivial component of the universal grading is the fusion subcategory $\mathcal{C}_{ad}$. Recall that $\mathcal{C}_{ad}$ is defined as the smallest fusion subcategory of $\mathcal{C}$ containing all $X \otimes X^*$ for each simple object $X \in \mathcal{O}(\mathcal{C})$ of $\mathcal{C}$. The universality of this grading consists in the fact that any other grading factors through the universal grading.

The following lemma appears in [8] and it will be needed in the proof of Theorem 2.1.

**Lemma 1.1.** *Let $\mathcal{C}$ be a fusion category and*

$$\mathcal{C} = \bigoplus_{g\in U_{\mathcal{C}}} \mathcal{C}_g$$

*be its universal grading. There is a one-to-one correspondence between fusion subcategories $\mathcal{D} \subset \mathcal{C}$ containing $\mathcal{C}_{ad}$ and subgroups $G \subset U_{\mathcal{C}}$, namely*

$$D \mapsto G_{\mathcal{D}} := \{g \in U_{\mathcal{C}} | \mathcal{D} \cap \mathcal{C}_g \neq 0\}$$

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*and*

$$G \mapsto \mathcal{D}_G := \bigoplus_{g \in G} \mathcal{C}_g.$$

### 1.4. The commutator subcategory of a fusion category

Recall the notion of the commutator subcategory from [10]. If $\mathcal{D}$ is a fusion subcategory of $\mathcal{C}$ then $\mathcal{D}^{co}$ is the full abelian subcategory of $\mathcal{C}$ generated by those objects $X$ such that $X \otimes X^* \in \mathcal{D}$. If $K_0(\mathcal{C})$ is a commutative ring then it is shown in [10] that $\mathcal{D}^{co}$ is a fusion subcategory of $\mathcal{C}$. In general the result is not true.

### 1.5. Definition of a normal fusion subcategory

The following notion of a normal tensor functor was introduced in [3]. If $\mathcal{C}$ and $\mathcal{E}$ are fusion categories then a tensor functor $F : \mathcal{C} \to \mathcal{E}$ is normal if the following property is satisfied: if $m_{\mathcal{C}}(1_{\mathcal{C}},\ F(X)) > 0$ for some simple object $X \in \mathrm{Irr}(\mathcal{C})$ then $F(X) = \mathrm{FPdim}(X)1_{\mathcal{C}}$. Recall the multiplicity form $m_{\mathcal{C}}$ defined on $K_0(\mathcal{C})$ by $m_{\mathcal{C}}([X], [Y]) = \delta_{[X],[Y]}$ for any two simple objects $X, Y \in \mathcal{O}(\mathcal{C})$.

The fusion subcategory of $\mathcal{C}$ generated by all objects $X \in \mathcal{C}$ with $F(X) = \mathrm{FPdim}(X)1_{\mathcal{C}}$ is called the kernel of $F$ and denoted by $\ker_{\mathcal{C}} F$. Recall from [3] that a fusion subcategory $\mathcal{D} \subset \mathcal{C}$ is called normal if there is a normal tensor functor $F : \mathcal{C} \to \mathcal{E}$ such that $\mathcal{D} = \ker_{\mathcal{C}} F$.

**Example 1.2.** Let $L$ be a normal Hopf subalgebra of a finite dimensional Hopf algebra $A$. It follows by [3, Proposition 3.9] that $\mathrm{Rep}(A//L)$ is a normal fusion subcategory of $\mathrm{Rep}(A)$. The converse of this fact is not true in general.

**Remark 1.3.** It is easy to check that if $\mathcal{D}$ is a normal fusion subcategory of $\mathcal{C}$ then $\mathcal{D} \cap \mathcal{C}_1$ is a normal fusion subcategory of $\mathcal{C}_1$ for any fusion subcategory $\mathcal{C}_1$ of $\mathcal{C}$.

### 1.6. On the commutator and the radical of a fusion subcategory

In this subsection we recall few notions and results from [2] that will be used in this paper. For a fusion subcategory $\mathcal{D}$ of a fusion category $\mathcal{C}$ define its radical as

$$\mathrm{rad}_{\mathcal{C}}(\mathcal{D}) = \{X \in \mathcal{O}(\mathcal{C}) | X^{\otimes n} \in \mathcal{D} \text{ for some } n \geq 1\}. \tag{1.1}$$

If $K_0(\mathcal{C})$ is commutative then clearly $\mathrm{rad}_{\mathcal{C}}(\mathcal{D})$ is a fusion subcategory.

Note that for any fusion category $\mathcal{D}$ one has $\mathrm{rad}_{\mathcal{C}}(\mathcal{D}) \subseteq \mathcal{D}^{co}$ since $X^*$ always appear as a constituent of a tensor power of $X$ (see also [21, Remark 3.2]). In [2] it is shown that for a normal fusion subcategory $\mathcal{D}$ the above two notions of radical and commutator coincide.

For a fusion category $\mathcal{C}$ denote by $\mathrm{Inv}(\mathcal{C})$, the set of all invertible objects (up to isomorphism) of $\mathcal{C}$. Suppose that $\mathcal{D}$ is a normal fusion subcategory of $\mathcal{C}$ fitting into an exact sequence

$$\mathcal{D} \to \mathcal{C} \xrightarrow{F} \mathcal{E} \tag{1.2}$$

of fusion categories. Then by [2, Proposition 5.3] it follows that $\mathcal{D}^{co}$ is a fusion subcategory of $\mathcal{C}$ and one has that

$$\mathcal{D}^{co} = \mathrm{rad}_{\mathcal{C}}(\mathcal{D}) = \{X \in \mathcal{C} | F(X) = \mathrm{FPdim}(X)M,\, M \in \mathrm{Inv}(\mathcal{E})\}. \tag{1.3}$$

**Example 1.4.** Let $L$ be a normal Hopf subalgebra of $A$. Then the commutator ideal $[A, L]$ is a Hopf ideal of $A$ and it is shown in [7, Theorem 2.2] that

$$\mathrm{Rep}(A//L)^{co} = \mathrm{Rep}(A/[A, L]). \tag{1.4}$$

The following corollary is well known. For the sake of completeness we include a sketch of its proof below.

**Corollary 1.5.** *Any group extension of* Vec *is a pointed fusion category.*

**Proof.** Let $\mathcal{C} = \oplus_{g \in G} \mathcal{C}_g$ be a graded fusion category with the trivial component $\mathcal{C}_1 = \mathrm{Vec}$. By [2, Proposition 5.4] it follows that $\mathcal{C}_g$ has up to isomorphism just one simple object $X_g$ which is also invertible. □

**Remark 1.6.** Note that if $\mathcal{C} = \oplus_{g \in G} \mathcal{C}_g$ is a graded fusion category then $\mathcal{C}_1^{co} = \mathrm{rad}_{\mathcal{C}}(\mathcal{C}_1) = \mathcal{C}$. Indeed any simple object $X \in \mathcal{C}_g$ satisfies that $X^{\otimes n} \in \mathcal{C}_{g^n} = \mathcal{C}_1$ if $n$ is the order of $G$. This shows that $\mathrm{rad}_{\mathcal{C}}(\mathcal{C}_1) = \mathcal{C}$. On the other hand since $X^* \in \mathcal{C}_{g^{-1}}$ it follows that $X \otimes X^* \in \mathcal{C}_1$ and therefore $\mathcal{C}_1^{co} = \mathcal{C}$.

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*1.7. Normal Hopf subalgebras and normal fusion subcategories of* $\mathrm{Rep}(A)$

Let $A$ be a finite dimensional semisimple Hopf algebra over $\mathbb{C}$. Then $A$ is also cosemisimple [14]. The character ring $C(A)$ of $A$ is a semisimple subalgebra of $A^*$ [26] and it has a vector space basis given by the set $\mathrm{Irr}(A)$ of irreducible characters of $A$. Moreover, $C(A) = \mathrm{Cocom}(A^*)$, the space of cocommutative elements of $A^*$. By duality, the character ring of $A^*$ is a semisimple subalgebra of $A$ and $C(A^*) = \mathrm{Cocom}(A)$. If $M$ is an $A$-representation with character $\chi$ then $M^*$ is also an $A$-representation with character $\chi^* = \chi \circ S$. This induces an involution " $*$ " $: C(A) \to C(A)$ on $C(A)$. Let $m_{_A}(\chi, \ \mu)$ be the multiplicity form on $C(A)$. We use the notation $G(A)$ for the set of grouplike elements of $A$. Let $t_{_A} \in A^*$ denote the integral of $A^*$ with $t_{_A}(1) = |A|$. It is known that $t_{_A}$ is also the regular character of $A$ [13, Proposition 4.1].

It is also well known that $\mathrm{Rep}(A)$ is a fusion category. Moreover there is a maximal central Hopf subalgebra $K(A)$ of $A$ such that $\mathrm{Rep}(A)_{ad} = \mathrm{Rep}(A//K(A))$, see [10]. Since $K(A)$ is commutative it follows that $K(A) = k[U_{_A}]^*$ where $U_{_A}$ is the universal grading group of $\mathrm{Rep}(A)$.

**Example 1.7.** If $A = kG$ then $K(A) = kZ(G)$ and $U_{_A} = \widehat{Z(G)}$, the linear dual group of the center $Z(G)$ of $G$.

Let $\mathcal{D}$ be a fusion subcategory of $\mathrm{Rep}(A)$ and $\mathcal{O}(\mathcal{D})$ be its set of simple objects up to isomorphisms. Then $I_{_\mathcal{D}} := \cap_{V\in\mathcal{O}(\mathcal{D})}\mathrm{Ann}_A(V)$ is a Hopf ideal in $A$ [23] and $\mathcal{D} = \mathrm{Rep}(A/I_{_\mathcal{D}})$. For a fusion subcategory $\mathcal{D} \subset \mathrm{Rep}(A)$ define its regular character as $r_{_\mathcal{D}} := \sum_{X\in\mathcal{O}(\mathcal{D})} \dim_k(X)\chi_X$ where $\mathrm{Irr}(\mathcal{D})$ is the set of irreducible objects of $\mathcal{D}$ and $\chi_X$ is the character of $X$ as an $A$-representation. Thus $r_{_\mathcal{D}} \in C(A)$.

For a finite dimensional $A$-representation $M$ with associated character $\chi \in C(A)$ recall [5] the definition of the Hopf subalgebra $A_\chi$, the kernel of the character $\chi$. It is the largest Hopf subalgebra of $A$ that acts trivially on $M$.

**Theorem 1.8.** *Let $A$ be a finite dimensional semisimple Hopf algebra and $\mathcal{D}$ be a fusion subcategory of* $\mathrm{Rep}(A)$*. Then $\mathcal{D} = \mathrm{Rep}(A//L)$ for some normal Hopf subalgebra $L$ of $A$ if and only if the regular character $r_{_\mathcal{D}}$ of $\mathcal{D}$ is central in $A^*$. In this case $L = A_{r_{_\mathcal{D}}}$, the kernel of $r_\mathcal{D}$.*

**Proof.** If $\mathcal{D} = \mathrm{Rep}(A//L)$ as above then by [5, Theorem 2.4] it follows that $r_{_\mathcal{D}}$ is an integral of $(A//L)^*$. Since this is a normal Hopf subalgebra of $A^*$, it follows from [16, Lemma 1] that $r_{_\mathcal{D}}$ is central in $A^*$. Conversely, if $r_{_\mathcal{D}}$ is central in $A^*$ then by [5, Proposition 3.3] it follows that the kernel $A_{r_{_\mathcal{D}}}$ is a normal Hopf subalgebra of $A$. Since $r_{_\mathcal{D}}^2 = \mathrm{FPdim}(\mathcal{D})r_{_\mathcal{D}}$ the same proposition loc. cit. implies that $\mathcal{D} = \mathrm{Rep}(A//A_{r_{_\mathcal{D}}})$. □

**Definition 1.9.** Let $L$ be a normal Hopf subalgebra of $A$. An irreducible character $\alpha$ of $L$ is called $A$-stable if there is a character $\chi \in \mathrm{Rep}(A)$ such that $\chi \downarrow_{_L}^{^A} = \frac{\chi(1)}{\alpha(1)}\alpha$. Such a character $\chi \in \mathrm{Irr}(A)$ is said to seat over the character $\alpha \in \mathrm{Irr}(L)$.

The set of all irreducible $A$-characters seating over $\alpha$ is denoted by $\mathrm{Irr}(A|_\alpha)$. Denote by $G_A^{st}(L)$ the set of all $A$-stable linear characters of $L$. Clearly $G_A^{st}(L)$ is a subgroup of the group of grouplike elements $G(L^*)$ of the dual Hopf algebra of $L$.

Theorem 4.5 and Example 1.4 imply the following.

**Lemma 1.10.** *Let $A$ be a semisimple Hopf algebra and $L$ be a normal Hopf subalgebra of $A$. Suppose that $M$ is an irreducible $A$-module affording the character $\chi$. Then $M \in \mathcal{O}(\mathrm{Rep}(A//L)^{co})$ if and only if $L$ acts trivially on some tensor power $M^{\otimes n}$ of $M$. In these conditions $\chi \downarrow_{_L}^{^A} = \chi(1)\psi$ for some $A$-stable linear character $\psi$ of $L$.*

**Proof.** By Example 1.2 $\mathrm{Rep}(A//L)$ is a normal fusion subcategory of $\mathrm{Rep}(A)$ and one has that $\mathrm{Rep}(A//L)^{co} = \mathrm{rad}(\mathrm{Rep}(A//L))$. This shows that $M \in \mathcal{O}(\mathrm{Rep}(A//L)^{co})$ if and only if $L$ acts trivially on some tensor power $M^{\otimes n}$ of $M$. If $M \in \mathrm{rad}(\mathrm{Rep}(A//L))$ then by Eq. (1.3) applied to the functor $F = \mathrm{res}_L^A$ it follows that $\chi \downarrow_{_L}^{^A} = \chi(1)\psi$ for some $A$-stable linear character $\psi$ of $L$. □

For any character $\chi$ of $A$ denote by $\mathrm{Irr}(\chi)$ the set of all irreducible constituents of $\chi$. For $\mathcal{C} \subset \mathrm{Rep}(A)$ we denote by $\mathcal{C}\chi$ the full abelian subcategory of $\mathrm{Rep}(A)$ generated by all the objects $a\chi$ with $a \in \mathcal{O}(\mathcal{C})$.

**Proposition 1.11.** *Suppose $A$ is a semisimple Hopf algebra and $L$ is a normal Hopf subalgebra of $A$. Moreover suppose that there is a fusion subcategory $\mathcal{C} \subset \mathrm{Rep}(A)$ with a faithful $G$-grading $\mathcal{C} = \oplus_{g\in G}\mathcal{C}_g$ whose trivial component is* $\mathrm{Rep}(A//L)$*. Then for any $\chi \in \mathcal{C}_g$ one has that $\chi \downarrow_{_L}^{^A} = \chi(1)\psi_g$ for some $A$-stable linear character $\psi_g$ of $L$. Moreover one has $\mathcal{O}(\mathcal{C}_g) = \mathrm{Irr}(A|_{\psi_g})$ and this set can be described as*

$$\mathcal{C}_g = \mathcal{C}_1\chi = \mathrm{Irr}(\psi_g \uparrow_{_L}^{^A}).$$

**Proof.** By Remark 1.6 it follows that $\chi \in \mathrm{Rep}(A//L)^{co}$. Therefore by Lemma 1.10 one has $\chi \downarrow_{_L}^{^A} = \chi(1)\psi_g$ for some $A$-stable linear character $\psi_g$ of $L$. On the other hand by Example 1.2 $\mathcal{C}_1 = \mathrm{Rep}(A//L)$ is a normal fusion subcategory of $\mathcal{C}$ and it follows by [2, Proposition 5.4] $\mathcal{C}_g = \mathcal{C}_1\chi$. The equality $\mathcal{C}_1\chi = \mathrm{Irr}(\psi_g \uparrow_{_L}^{^A})$ follows then by Frobenius reciprocity. Note that the last two equalities can also be deduced from Theorem 4.3 of [4]. □

## 2. Fusion subcategories of a Deligne direct product of categories

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two fusion categories and form the Deligne product $\mathcal{C} := \mathcal{C}_1 \boxtimes \mathcal{C}_2$ . Identify $\mathcal{C}_1$ and $\mathcal{C}_2$ to $\mathcal{C}_1 \boxtimes 1$ and $1 \boxtimes \mathcal{C}_2$ respectively as fusion subcategories of $\mathcal{C}$. Recall that every simple object of $\mathcal{C}$ is of the type $X_1 \boxtimes X_2$ where $X_i$ is a simple object of $\mathcal{C}_i$.

Let $\mathcal{D} \subset \mathcal{C}$ be any fusion subcategory. Define $\mathcal{L}^i(\mathcal{D}) := \mathcal{D} \cap \mathcal{C}_i$, with $i = 1, 2$. Let also $\mathcal{K}^1(\mathcal{D})$ be the fusion subcategory generated by all simple objects $X_1$ of $\mathcal{C}_1$ such that $X_1 \boxtimes X_2 \in \mathcal{D}$ for some simple object $X_2$ of $\mathcal{C}_2$. Similarly define the fusion subcategory $\mathcal{K}^2(\mathcal{D})$. Clearly $\mathcal{L}^1(\mathcal{D}) \boxtimes \mathcal{L}^2(\mathcal{D}) \subset \mathcal{D} \subset \mathcal{K}^1(\mathcal{D}) \boxtimes \mathcal{K}^1(\mathcal{D})$. The following theorem is an analogue of Goursat's lemma for the direct product of groups and it appears in [8]. For the sake of completeness we include its proof below.

**Theorem 2.1.** *Let $\mathcal{D} \subset \mathcal{C}_1 \boxtimes \mathcal{C}_2$ be a fusion subcategory. Then there are a finite group $G$ and faithful $G$-gradings $\mathcal{K}^i(\mathcal{D}) = \bigoplus_{x\in G} \mathcal{K}^i(\mathcal{D})_x$ with trivial components $\mathcal{L}^i(\mathcal{D})$, $i = 1, 2$ such that*

$$\mathcal{D} = \bigoplus_{x\in G} \mathcal{K}^1(\mathcal{D})_x \boxtimes \mathcal{K}^2(\mathcal{D})_x. \tag{2.1}$$

**Proof.** First we will show that

$$\mathcal{D} \supset \mathcal{K}^1(\mathcal{D})_{ad} \boxtimes \mathcal{K}^2(\mathcal{D})_{ad} = (\mathcal{K}^1(\mathcal{D}) \boxtimes \mathcal{K}^2(\mathcal{D}))_{ad}, \tag{2.2}$$

where as above $\mathcal{K}^i(\mathcal{D})_{ad}$ denotes the adjoint subcategory of $\mathcal{K}^i(\mathcal{D})$. Note that if $X_1 \boxtimes X_2 \in \mathcal{O}(\mathcal{D})$ then $(X_1 \otimes X_1^*) \boxtimes (X_2 \otimes X_2^*) \in\in \mathcal{O}(\mathcal{D})$. Since $X_i \otimes X_i^*$ contains the unit object of $\mathcal{C}_i$ it follows that $(X_1 \otimes X_1^*) \boxtimes 1 \in \mathcal{D}$ and $1 \boxtimes (X_2 \otimes X_2^*) \in \mathcal{D}$. Therefore $\mathcal{K}^i(\mathcal{D})_{ad} \subset \mathcal{L}^i(\mathcal{D})$ for $i = 1, 2$. Since $\mathcal{L}^1(\mathcal{D}) \boxtimes \mathcal{L}^2(\mathcal{D}) \subset \mathcal{D}$, it follows that the above inclusion implies Eq. (2.2). Now let $U_{\mathcal{K}^i(\mathcal{D})}$ be the universal grading group of $\mathcal{K}^i(\mathcal{D})$. Lemma 1.1 applied to the inclusion Equation (2.2) gives that

$$\mathcal{D} = \bigoplus_{\gamma\in\Gamma} (\mathcal{K}^1(\mathcal{D}) \boxtimes \mathcal{K}^2(\mathcal{D}))_\gamma$$

for some subgroup $\Gamma \in U_{\mathcal{K}^1(\mathcal{D})\boxtimes\mathcal{K}^2(\mathcal{D})} = U_{\mathcal{K}^1(\mathcal{D})} \times U_{\mathcal{K}^2(\mathcal{D})}$. On the other hand it follows from the definition of $\mathcal{K}^i(\mathcal{D})$ that the maps $\Gamma \to U_{\mathcal{K}^1(\mathcal{D})}$ and $\Gamma \to U_{\mathcal{K}^2(\mathcal{D})}$ are surjective. Then Goursat's lemma for groups (see [25]) implies that $\Gamma$ equals the fiber product $U_{\mathcal{K}^1(\mathcal{D})} \times_G U_{\mathcal{K}^2(\mathcal{D})}$ for some group $G$ equipped with group epimorphisms $U_{\mathcal{K}^i(\mathcal{D})} \to G$, $i = 1, 2$. Then these epimorphisms define faithful $G$-gradings of $\mathcal{K}^i(\mathcal{D})$ such that Eq. (2.1) holds. In this case note that $\mathcal{L}^i(\mathcal{D}) := \mathcal{D} \cap \mathcal{C}^i$ equals the trivial component of the $G$-grading of $\mathcal{K}^i(\mathcal{D})$ for all $1 \le i \le 2$. □

**Remark 2.2.** By Remark 1.6 it follows that in this situation one has

$$\mathrm{rad}(\mathcal{L}^i(\mathcal{D})) \supseteq \mathcal{L}^i(\mathcal{D})^{co} \supseteq \mathcal{K}^i(\mathcal{D})$$

for all $1 \le i \le 2$.

**Proposition 2.3.** *Let $\mathcal{D}$ be any fusion subcategory of $\mathcal{C}_1 \boxtimes \mathcal{C}_2$. Then*

$$\mathcal{D}^{co} = \mathcal{L}^1(\mathcal{D})^{co} \boxtimes \mathcal{L}^2(\mathcal{D})^{co} \tag{2.3}$$

*and*

$$\mathrm{rad}(\mathcal{D}) = \mathrm{rad}(\mathcal{L}^1(\mathcal{D})) \boxtimes \mathrm{rad}(\mathcal{L}^2(\mathcal{D})). \tag{2.4}$$

**Proof.** Suppose that $X \boxtimes Y \in \mathcal{D}^{co}$. Then $(X \otimes X^*) \boxtimes (Y \otimes Y^*) \in \mathcal{D}$ and therefore $X \otimes X^* \in \mathcal{L}^1(\mathcal{D})$ since $1_{\mathcal{C}_2}$ is a constituent of $Y \otimes Y^*$. Similarly $Y \otimes Y^* \in \mathcal{L}^2(\mathcal{D})$. Thus $\mathcal{D}^{co} \subset \mathcal{L}^1(\mathcal{D})^{co} \boxtimes \mathcal{L}^2(\mathcal{D})^{co}$. On the other hand, clearly $\mathcal{L}^1(\mathcal{D})^{co} \boxtimes \mathcal{L}^2(\mathcal{D})^{co} \subset \mathcal{D}^{co}$.

Suppose now that $X \boxtimes Y \in \mathrm{rad}(\mathcal{D})$. Then $X^n \boxtimes Y^n \in \mathcal{D}$ for some $n \geq 1$. As in [21, Remark 3.2] there is $m \geq 1$ such that $Y^{mn} = (Y^n)^m$ contains the unit element of $\mathcal{C}_2$. This implies that $X^{mn} \in \mathcal{L}^1(\mathcal{D})$ and therefore $X \in \mathrm{rad}(\mathcal{L}^1(\mathcal{D}))$. Similarly $Y \in \mathrm{rad}(\mathcal{L}^2(\mathcal{D}))$ which shows that $\mathrm{rad}(\mathcal{D}) \supset \mathrm{rad}(\mathcal{L}^1(\mathcal{D})) \boxtimes \mathrm{rad}(\mathcal{L}^2(\mathcal{D}))$. The other inclusion is immediate. □

**Theorem 2.4.** *Any fusion subcategory of $\mathcal{C}_1 \boxtimes \mathcal{C}_2$ with trivial intersections with $\mathcal{C}_1$ and $\mathcal{C}_2$ is pointed.*

**Proof.** With the above notations it follows that $\mathcal{L}^1(\mathcal{D}) = \mathrm{Vec}$ and $\mathcal{L}^2(\mathcal{D}) = \mathrm{Vec}$. Thus $\mathcal{K}^i(\mathcal{D})$ are group extensions of Vec and by Corollary 1.5 they are pointed fusion subcategories. It follows that $\mathcal{D}$ is also a pointed fusion category. □

### *2.1. Normal fusion subcategories of a Deligne tensor product*

Suppose now that $\mathcal{D}$ is a normal fusion subcategory of $\mathcal{C} := \mathcal{C}_1 \boxtimes \mathcal{C}_2$. Applying Theorem 2.1 one obtains

$$\mathcal{D} = \oplus_{x\in G} \mathcal{K}^1(\mathcal{D})_x \boxtimes \mathcal{K}^2(\mathcal{D})_x \tag{2.5}$$

where $\mathcal{K}^i(\mathcal{D})$ are $G$-faithful graded fusion subcategories of $\mathcal{C}_i$ with

$$\mathcal{K}^i(\mathcal{D}) = \oplus_{x\in G} \mathcal{K}^i(\mathcal{D})_x. \tag{2.6}$$

Moreover by Remark 1.3 one has that $\mathcal{L}^i(\mathcal{D})$ are normal fusion subcategories of $\mathcal{C}_i$ and $\mathrm{rad}(\mathcal{L}^i(\mathcal{D})) = \mathcal{L}^i(\mathcal{D})^{co} \supseteq \mathcal{K}^i(\mathcal{D})$ for all $1 \le i \le 2$.

**Remark 2.5.** Since $\mathcal{L}^i(\mathcal{D})$ is a normal fusion subcategory of $\mathcal{K}^i(\mathcal{D})$ it follows by [2, Proposition 5.4] that $\mathcal{K}^i(\mathcal{D})_x = \mathcal{L}^i(\mathcal{D})a_x^i$ for any simple object $a_x^i \in \mathcal{K}^i(\mathcal{D})_x$.

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## 3. Hopf subalgebras of semisimple Drinfeld doubles

Recall that the Drinfeld double $D(A)$ of a Hopf algebra $A$ is defined by $D(A) \cong A^{*\mathrm{cop}} \otimes A$ as coalgebras with the multiplication given by

$$(g \bowtie h)(f \bowtie l) = \sum g(h_1 \rightharpoonup f \leftharpoonup S^{-1}h_3) \bowtie h_2 l, \tag{3.1}$$

for all $h, l \in A$ and $f, g \in A^*$. Moreover its antipode is given by $S(f \bowtie h) = S^{-1}(h)S(f)$. It is well known that $D(A)$ is a semisimple Hopf algebra if and only if $A$ is a semisimple Hopf algebra [18]. It is also known that $D(A)$ is a cocycle twist of $A^{*\mathrm{cop}} \otimes A$ and therefore $\mathrm{Rep}(D(A)^*) = \mathrm{Rep}(A^{\mathrm{op}}) \boxtimes \mathrm{Rep}(A^*)$.

### 3.1. On Hopf subalgebras of Drinfeld doubles $D(A)$

If $H$ is a Hopf subalgebra of $D(A)$ then clearly one has the following inclusion

$$\mathrm{Rep}(H^*) \subset \mathrm{Rep}(D(A)^*) = \mathrm{Rep}(A^{\mathrm{op}}) \boxtimes \mathrm{Rep}(A^*).$$

Let $\mathcal{C}_1 := \mathrm{Rep}(A^{\mathrm{op}})$, $\mathcal{C}_2 := \mathrm{Rep}(A^*)$ and $\mathcal{D} := \mathrm{Rep}(H^*)$. Theorem 2.1 implies that $\mathrm{Rep}(H^*) = \bigoplus_{x\in G} \mathcal{K}^1(\mathcal{D})_x \boxtimes \mathcal{K}^2(\mathcal{D})_x$. Moreover one has the inclusions $\mathcal{K}^1(\mathcal{D})_{ad} \subset \mathcal{L}^1(\mathcal{D}) \subset \mathcal{K}^1(\mathcal{D}) \subset \mathrm{Rep}(A^{\mathrm{op}})$, and $\mathcal{K}^2(\mathcal{D})_{ad} \subset \mathcal{L}^2(\mathcal{D}) \subset \mathcal{K}^2(\mathcal{D}) \subset \mathrm{Rep}(A^*)$ with the faithful $G$-gradings $\mathcal{K}^i(\mathcal{D}) = \bigoplus_{x\in G}(\mathcal{K}^i(\mathcal{D}))_x$. Note that in this situation $\mathcal{K}^i(\mathcal{D})_1 = \mathcal{L}^i(\mathcal{D})$ for all $i = 1, 2$. Also one can write $\mathcal{L}^1(\mathcal{D}) = \mathrm{Rep}(M_1^{*\mathrm{op}})$ where $M_1 := H \cap A^*$ is a Hopf subalgebra of $A^*$. Similarly $\mathcal{L}^2(\mathcal{D}) = \mathrm{Rep}(M_2^*)$ where $M_2 := H \cap A$ is a Hopf subalgebra of $A$.

**Theorem 3.1.** *Any Hopf subalgebra of $D(A)$ with trivial intersections with $A$ and $A^*$ is a group algebra.*

**Proof.** Let $H$ be a such Hopf subalgebra of $D(A)$. It follows that $\mathrm{Rep}(H^*)$ regarded as a fusion subcategory of $\mathrm{Rep}(A^{\mathrm{op}}) \boxtimes \mathrm{Rep}(A^*)$ as above has trivial intersections with both $\mathrm{Rep}(A^{\mathrm{op}})$ and $\mathrm{Rep}(A^*)$. Then Theorem 2.4 implies that $\mathrm{Rep}(H^*)$ is pointed. Thus $H^* = kG^*$ and $H = kG$. □

For any right $A$-comodule $M$ with comodule structure $\rho : M \to A \otimes M$ denote by $C_M$ the subcoalgebra of coefficients of $M$. Recall that $C_M$ is the smallest subcoalgebra $C \subset A$ with the property that $\rho(M) \subset C \otimes M$ [13]. If $d \in C(A^*)$ is the character of $M$ as an $A$-comodule then $C_M$ is also denoted by $C_d$.

By duality any left $A$-module $V$ with associated character $\chi \in C(A)$ can be regarded as a right $A^*$-module and one can associate to it as above its subcoalgebra of coefficients $C_\chi \subset A^*$.

Recall from [22] that a subset $X \subset \mathrm{Irr}(H^*)$ is closed under multiplication if for every two elements $c, d \in X$ in the decomposition of the product $cd = \sum_{e\in \mathrm{Irr}(H^*)} m_{c,d}^e e$ one has $e \in X$ whenever $m_e \neq 0$. Also a subset $X \subset \mathrm{Irr}(H^*)$ is closed under " $*$ " if $x^* \in X$ for all $x \in X$.

**Remark 3.2.** Following [22, Theorem 6] it follows that any subset $X \subset \mathrm{Irr}(H^*)$ closed under multiplication and " $*$ " generates a Hopf subalgebra $H(X)$ of $H$ defined by

$$H(X) := \oplus_{x\in X} C_x. \tag{3.2}$$

### 3.2. On a class of Hopf subalgebras of $D(A)$

Suppose that $L_1$ and $L_2$ are two normal Hopf subalgebras of $A$ and let $G$ be a finite group that can be simultaneously embedded in $G_A^{st}(L_1)$ and $G_{A^*}^{st}((A//L_2)^*)$ via the embeddings $\psi_1 : G \hookrightarrow G_A^{st}(L_1)$ and respectively $\psi_2 : G \hookrightarrow G_{A^*}^{st}((A//L_2)^*)$. Let $B(L_1, L_2, G, \psi_1, \psi_2)$ be the subcoalgebra of $D(A)$ defined by

$$B(L_1, L_2, G, \psi_1, \psi_2) = \bigoplus_{x\in G} C_{\psi_1(x)\uparrow_{L_1}^{A}} \bowtie C_{\psi_2(x)\uparrow_{(A//L_2)^*}^{A^*}}. \tag{3.3}$$

**Proposition 3.3.** *Suppose that $L_1$ and $L_2$ are two normal Hopf subalgebras of $A$ and $G$ is a finite group as above. Then $B(L_1, L_2, G, \psi_1, \psi_2)$ is a Hopf subalgebra of $D(A)$.*

**Proof.** By Remark 3.2 it is enough to show that the irreducible comodule characters of $B(L_1, L_2, G, \psi_1, \psi_2)$ form a set closed under multiplication and duality. Note that an irreducible comodule character of the Hopf subalgebra $B(L_1, L_2, G, \psi_1, \psi_2)$ is of the type $\chi \bowtie d$ with $\chi \in \mathrm{Irr}(A)$ seating over $\psi_1(x)$ (i.e. $\chi \in \mathrm{Irr}(A|_{\psi_1(x)})$) and $d \in \mathrm{Irr}(A^*)$ seating over $\psi_2(x)$ (i.e. $d \in \mathrm{Irr}(A^*|_{\psi_2(x)})$), for some $x \in G$. Suppose moreover that $\chi' \in \mathrm{Irr}(A)$ seats over $\psi_1(y)$ and $d' \in \mathrm{Irr}(A^*)$ seats over $\psi_2(y)$ for some other $y \in G$. It follows that for the product in $\mathrm{Rep}(D(A)^*)$ one has

$$(\chi \bowtie d)(\chi' \bowtie d') = \chi\chi' \bowtie dd' \tag{3.4}$$

and the irreducible constituents of $\chi\chi'$ are all seating over $\psi_1(xy)$ while the irreducible constituents of $dd'$ are all seating over $\psi_2(xy)$. This shows that the above set is closed under product. Moreover the set is closed under duality since $\chi^* \in \mathrm{Irr}(A|_{\psi_1(x^{-1})})$ and $d^* \in \mathrm{Irr}(A^*|_{\psi_2(x^{-1})})$. □

Note that if $G$, $L_1$ and $L_2$ are as above then one has

$$B(L_1, L_2, G, \psi_1, \psi_2) \cap A = L_2 \quad \text{and} \quad B(L_1, L_2, G, \psi_1, \psi_2) \cap A^* = (A//L_1)^*. \tag{3.5}$$

If $G = 1$ then $\psi_1$ and $\psi_2$ are trivial and we denote the Hopf subalgebra $B(L_1, L_2, \{1\}, 1, 1)$ simply by $B(L_1, L_2)$.

### 3.3. Hopf subalgebras of $D(A)$ with normal intersections with $A$ and $A^*$

**Theorem 3.4.** *Let $A$ be a semisimple Hopf algebra. Then any Hopf subalgebra $H$ of $D(A)$ with normal Hopf subalgebra intersections with $A$ and $A^*$ is of the form $B(L_1, L_2, G, \psi_1, \psi_2)$ defined above.*

**Proof.** Let $\mathcal{D} := \mathrm{Rep}(H^*) \subset \mathrm{Rep}(A^{\mathrm{op}}) \boxtimes \mathrm{Rep}(A^*)$. Using the notations from Section 3.1 it follows that $M_1$ and $M_2$ are normal Hopf subalgebras of $A^*$ and respectively $A$. One may suppose that $M_1 := (A//L_1)^*$ for some normal Hopf subalgebra $L_1$ of $A$. Thus $\mathcal{L}^1(\mathcal{D}) = \mathrm{Rep}((A//L_1)^{\mathrm{op}})$. Similarly, write $\mathcal{L}^2(\mathcal{D}) = \mathrm{Rep}(L_2^*)$ with $L_2 := M_2$ a normal Hopf subalgebra of $A$. Suppose that $\eta \bowtie d \in \mathcal{K}^1(\mathcal{D})_x \boxtimes \mathcal{K}^2(\mathcal{D})_x \subset \mathcal{D}$ for a given $x \in G$. Then Proposition 1.11 implies that $\eta \downarrow_{L_1}^{A} = \eta(1)f_x$ for some $A$-stable linear character $f_x$ of $L_1$. This shows that $G$ can be regarded as a subgroup of $G_A^{st}(L_1)$ via the map $\psi_1$ given by $x \mapsto f_x$. By duality, the same argument applied for $\mathcal{K}^2(\mathcal{D})_x$ gives that $d \downarrow_{(A//L_2)^*}^{A^*} = \epsilon(d)g_x$ for some $A^*$-stable linear character $g_x$ of $(A//L_2)^*$. Therefore $G$ can also be identified with a subgroup of $G_{A^*}^{st}((A//L_2)^*)$ via the map $\psi_2$ given by $x \mapsto g_x$. Moreover by the same Proposition 1.11 the set of simple objects of $\mathcal{K}^1(\mathcal{D})_x$ can be identified with the set of irreducible modules of $\psi_1(x) \uparrow_{L_1}^{A}$. A similar result holds for $\mathcal{K}^2(\mathcal{D})_x$ and the proof of the theorem is complete. □

**Corollary 3.5.** *Any normal Hopf subalgebra of $D(A)$ is of the type $B(L_1, L_2, G, \psi_1, \psi_2)$ defined above.*

**Corollary 3.6.** *With the notations from Theorem 3.4 it follows that*

$$\mathrm{Rep}(H^*) \subset \mathrm{Rep}(A//L_1)^{co} \boxtimes \mathrm{Rep}(L_2^*)^{co}.$$

**Proof.** It follows directly from above by applying Lemma 1.10. □

## 4. Normal Hopf subalgebras of Drinfeld doubles $D(A)$

In this section we give some necessary conditions to be satisfied by the Hopf subalgebras $L_1$ and $L_2$ in order for $B(L_1, L_2, G, \psi_1, \psi_2)$ to be a normal Hopf subalgebra of $D(A)$. In the case of $B(L_1, L_2)$ we show that these conditions are also sufficient.

### 4.1. Commutativity between $L_1$ and $L_2$

First we need the following lemma.

**Lemma 4.1.** *Suppose that $K$ is a Hopf subalgebra of a semisimple Hopf algebra $H$ and $x \in H$. Then $Sm_2xm_1 = \epsilon(m)x$ for all $m \in K$ if and only if $mx = xm$ for all $m \in K$.*

**Proof.** One has that $xm = m_1S(m_2)xm_3 = m_1\epsilon(m_2)x = mx$. The converse also follows since $Sm_2xm_1 = Sm_2m_1x = \epsilon(m)x$. □

**Remark 4.2.** Suppose that $K$ is a normal Hopf subalgebra of a Hopf algebra $H$ and let $A = H//K$ be the quotient Hopf algebra of $H$ via $\pi_K : H \to H//K$. Then since $\pi^* : A^* \to H^*$ is an injective Hopf algebra map it follows that $\pi^*(A^*)$ is a Hopf subalgebra of $H^*$. Under this identification it can be checked that

$$\pi^*(A^*) = \{f \in H^* | f(hm) = f(h)\epsilon(m);\ h \in H, m \in K\} \tag{4.1}$$

is the set of linear functionals invariant under left (right) $K$-translations.

For two Hopf subalgebras $L_1, L_2$ of $A$ denote as usually by $[L_1, L_2]$ the commutator ideal generated by $ab - ba$ with $a \in L_1$ and $b \in L_2$. Note that two Hopf subalgebras $L_1$ and $L_2$ of $A$ commute elementwise if and only if $[L_1, L_2] = 0$.

Let $K$ be a Hopf subalgebra of $A$ and $\Lambda_K \in A$ be its idempotent integral. Then it is well known (see [5]) that the induced representation $A \otimes_K k$ is isomorphic to $A\Lambda_K$ via the map $a \otimes_K 1 \mapsto a\Lambda_K$.

Note that if $K$ is a normal Hopf subalgebra of $A$ then $\Lambda_K$ is a central element of $A$ by [16, Lemma 1]. Then $K$ acts trivially on the induced module $A \otimes_K k$. Indeed, since the induced module is isomorphic to $A\Lambda_K$ one has that $ma\Lambda_K = m\Lambda_K a = \epsilon(m)a\Lambda_K$ for all $m \in K$ and $a \in A$. Corollary 2.5 of [5] shows that $K$ is a normal Hopf subalgebra of $A$ if and only if $K$ acts trivially on the induced module $A \otimes_K k$.

**Theorem 4.3.** *Let $A$ be a semisimple Hopf algebra and $L_1, L_2$ be two normal Hopf subalgebras of $A$. If $B(L_1, L_2, G, \psi_1, \psi_2)$ is a normal Hopf subalgebra of $D(A)$ then the following relations $[L_1, L_2] = 0$ and $[(A//L_1)^*, (A//L_2)^*] = 0$ hold.*

**Proof.** Let $K := B(L_1, L_2, G, \psi_1, \psi_2)$. If $K$ is a normal Hopf subalgebra of $D(A)$ then as above, by [5, Corollary 2.5 ] one has $m((f \bowtie b) \otimes_K 1) = \epsilon(m)((f \bowtie b) \otimes_K 1)$ for all $m \in K$ and any $f \in A^*, b \in A$. In particular if $m \in L_2$ then one has that

$$\begin{aligned} m((f \bowtie b) \otimes_K 1) &= f_1(Sm_3)f_3(m_1)f_2 \bowtie m_2 b \otimes_K 1 \\ &= f_1(Sm_3)f_3(m_1)f_2 \bowtie b_1(Sb_2 m_2 b_3) \otimes_K 1 \\ &= f_1(Sm_3)f_3(m_1)f_2 \bowtie b_1 \otimes_K (Sb_2 m_2 b_3)1 \\ &= f_1(Sm_2)f_3(m_1)f_2 \bowtie b \otimes_K 1. \end{aligned}$$

This implies that $f_1(Sm_2)f_3(m_1)f_2 \bowtie b \otimes_K 1 = \epsilon(m)f \bowtie b \otimes_K 1$ and the previous remark gives that

$$(f_1(Sm_2)f_3(m_1)f_2 \bowtie b)\Lambda_K = \epsilon(m)(f \bowtie b)\Lambda_K. \tag{4.2}$$

Note that the integral of $K$ has the form

$$\Lambda_K = \sum_{x \in G} \psi_1(x) \uparrow_{L_1}^{A} \bowtie \psi_2(x) \uparrow_{(A//L_2)^*}^{A^*}$$

for all $f \in A^*$ and $b \in A$.

Since for $x \neq y$ one has that $C_{\psi_2(x)\uparrow_{(A//L_2)^*}^{A^*}} \cap C_{\psi_2(y)\uparrow_{(A//L_2)^*}^{A^*}} = 0$ there is a functional $g \in A^*$ such that $g = \epsilon$ on the coalgebra $L_2 = C_{\psi_2(1)\uparrow_{(A//L_2)^*}^{A^*}}$ and $g$ is zero on all the other coalgebras $C_{\psi_2(x)\uparrow_{(A//L_2)^*}^{A^*}}$ with $x \neq 1$.

Then put $b = 1$ in Eq. (4.2) and apply $\mathrm{id} \otimes g$ on both terms. One obtains that $f_1(Sm_2)f_3(m_1)f_2 t_{A//L_1} = \epsilon(m)ft_{A//L_1}$ for all $f \in A^*$. Evaluating both sides of the above equality at any $l \in L_1$ it follows that $Sm_2 l m_1 = \epsilon(l)m$ since $t_{(A//L_1)}(l) = \epsilon(l)$. Then Lemma 4.1 shows that $lm = ml$ for all $m \in L_2$ and any $l \in L_1$. Thus $L_1$ and $L_2$ commute elementwise.

It is known that $D(A) \cong D(A^{*\,op\,\mathrm{cop}})^{\mathrm{op}}$ as Hopf algebras via $f \bowtie a \mapsto a \bowtie f$ (see for instance [24, Theorem 3]). Note that under the above isomorphism $B(L_1, L_2, G, \psi_1, \psi_2) \cong B((A//L_2)^*, (A//L_1)^*, G, \psi_2, \psi_1)$ as a Hopf subalgebra of $D(A^{*\,op\,\mathrm{cop}})$. Indeed note that $(A//L_1)^* \bowtie L_2$ is sent to $(A^*//(A//L_2)^*)^* \bowtie (A//L_1)^*$ inside $D(A^{*\,op\,\mathrm{cop}})$.

Then the above argument of this proof applied to $A^{*\,op\,\mathrm{cop}}$ implies that if $B((A//L_2)^*, (A//L_1)^*, G, \psi_2, \psi_1)$ the normal Hopf subalgebra of the double $D(A^{*\,op\,\mathrm{cop}})$ then $[(A//L_2)^*,\ (A//L_1)^*] = 0$. □

### *4.2. Normal Hopf subalgebras of type $B(K, L)$*

One more preliminary lemma is needed.

**Lemma 4.4.** *Let $K$ be a normal Hopf subalgebra of a semisimple Hopf algebra $A$. Then $m(g(a_2)Sa_3a_1) = (g(a_2)Sa_3a_1)m$ for all $m \in K, g \in (A//M)^*$ and any $a \in A$.*

**Proof.** Since $(A//K)^*$ is a normal Hopf subalgebra of $A^*$ it follows that $Sf_1gf_2 \in (A//K)^*$ for all $f \in A^*$. Eq. (4.1) implies

$$\langle Sf_1gf_2, am\rangle = \epsilon(m)\langle Sf_1gf_2, a\rangle$$

for all $a \in A$ and all $m \in K$. It is easy to see that the above equality can be written as $f(Sm_1Sa_1g(a_2m_2)a_3m_3) = \epsilon(m)f(Sa_1g(a_2)a_3)$. Since $f$ was chosen arbitrary and $g \in (A//K)^*$ it follows that

$$Sm_1(Sa_1g(a_2)a_3)m_2 = \epsilon(m)(Sa_1g(a_2)a_3)$$

for all $a \in A$ and all $m \in K$. Applying the previous lemma it follows that

$$m(Sa_1g(a_2)a_3) = m(Sa_1g(a_2)a_3)$$

for all $a \in A$ and any $m \in K$. Now the result follows by passing to $K^{\mathrm{cop}} \subset A^{\mathrm{cop}}$ as a normal Hopf subalgebra. □

We now can prove the following theorem.

**Theorem 4.5.** *If $K$ and $L$ are normal Hopf subalgebras of $A$ then $B(K, L)$ is a normal Hopf subalgebra of $D(A)$ if and only if $[K,\ L] = 0$ and $[(A//K)^*,\ (A//L)^*] = 0$.*

**Proof.** Note that $B(K, L) := (A//K)^{*\mathrm{cop}} \bowtie L$. If $B(K, L)$ is a normal Hopf subalgebra of $D(A)$ then by Theorem 4.3 it follows that the two commutator ideals vanish.

Conversely, suppose that $[K,\ L] = 0$ and $[(A//K)^*,\ (A//L)^*] = 0$.

It will be shown first that if $[K,\ L] = 0$ then the Hopf subalgebra $B(K, L)$ is closed under the adjoint action of $A^{*\mathrm{cop}}$ regarded as a Hopf subalgebra of $D(A)$. Note that under the adjoint action of $A^{*\mathrm{cop}}$ one has that

$$f \cdot (g \bowtie l) = f_2(g \bowtie l)Sf_1 = f_2 g(l_1 \rightharpoonup Sf_1 \leftharpoonup Sl_3) \bowtie l_2,$$

for all $f \in A^{*\mathrm{cop}}, g \in (A//K)^{*\mathrm{cop}}$ and $l \in L$.

Thus in order to show that $f \cdot (g \bowtie l) \in B(K, L)$ it is enough to show that $f_2 g(a \rightharpoonup Sf_1 \leftharpoonup b) \in (A//K)^*$ for all $a, b \in L$.

By Eq. (4.1) $f_2 g(a \rightharpoonup Sf_1 \leftharpoonup b) \in (A//K)^{*cop}$ if and only if

$$\langle f_2 g(a \rightharpoonup Sf_1 \leftharpoonup b), cm\rangle = \epsilon(m)\langle f_2 g(a \rightharpoonup Sf_1 \leftharpoonup b), c\rangle$$

for all $f \in A^{*cop}$, $g \in (A//K)^{*cop}$, $c \in A$ and $m \in K$. Thus one has to show that

$$\langle f, SaSm_3Sc_3Sbc_1m_1g(c_2m_2)\rangle = \epsilon(m)\langle f, SaSc_3Sbc_1g(c_2)\rangle$$

for all $f \in A^*$, $g \in (A//K)^{*cop}$, $c \in A$, $a, b \in L$ and $m \in K$. Since $f \in A^*$ is arbitrary this is equivalent to $SaSm_3Sc_3Sbc_1m_1g(c_2m_2) = \epsilon(m)SaSc_3Sbc_1g(c_2)$. Therefore it is enough to show that

$$Sm_3Sc_3Sbc_1m_1g(c_2m_2) = \epsilon(m)Sc_3Sbc_1g(c_2) \tag{4.3}$$

for all $g \in (A//K)^{*cop}$, $c \in A$, $b \in L$ and $m \in K$. On the other hand since $g \in (A//K)^{*cop}$ the above equation can also be written as

$$Sm_2[Sc_3Sbc_1g(c_2)]m_1 = \epsilon(m)[Sc_3Sbc_1g(c_2)] \tag{4.4}$$

for all $g \in (A//K)^{*cop}$, $c \in A$, $b \in L$ and $m \in K$. Thus applying Lemma 4.1 for $x = Sc_3Sbc_1g(c_2)$ the above equation is equivalent to $m[Sc_3Sbc_1g(c_2)] = [Sc_3Sbc_1g(c_2)]m$ for all $a \in A$ and $x, y \in L$.

On the other hand note that applying Lemma 4.4 one has

$$\begin{aligned} m[Sc_3Sbc_1g(c_2)] &= m[(Sc_5Sbc_4)(Sc_3g(c_2)c_1)] \\ &= [(Sc_5Sbc_4)m(Sc_3g(c_2)c_1)] \\ &= Sy(Sa_5xa_4)k(Sa_3g(a_2)a_1) \\ &= [Sc_3Sbc_1g(c_2)]m \end{aligned}$$

which shows that Eq. (4.4) is satisfied.

One has $D(A) \cong D(A^{*\,op\,cop})^{op}$ as Hopf algebras via $f \bowtie a \mapsto a \bowtie f$ by [24, Theorem 3]. Then as in the proof of the previous Theorem, it follows by the same argument applied to $A^{*\,op\,cop}$ that $(A//K)^* \bowtie L$ is closed under the adjoint action of $A$ if and only if $[(A//K)^*, (A//L)^*] = 0$. □

**Corollary 4.6.** *With the above notations suppose that $B(L_1, L_2, G)$ is a normal Hopf subalgebra of $D(A)$. Then $B(L_1, L_2)$ is also a normal Hopf subalgebra of $D(A)$.*

**Proof.** It follows by Theorem 4.5 that the above conditions are satisfied. □

### *4.3. Bicrossed product of Hopf algebras*

Recall the notion of the bicrossed product of two Hopf algebras. Let $H$ be a Hopf algebra and $A, L$ be two Hopf subalgebras of $H$. We say that $(A, L)$ is a factorization of $H$ if the multiplication map $m : A \otimes L \to H$ is bijective. Note that in this case $A \cap L = k$ and by [15, Theorem 2.7.3] it follows that $H$ is a bicrossed product Hopf algebra of $A$ and $L$. It is also known that $D(A)$ is a bicrossed product of $A^{*cop}$ and $A$, see e.g. [15].

**Proposition 4.7.** *Let $A$ be a semisimple Hopf algebra and $B(L_1, L_2)$ a normal Hopf subalgebra of $D(A)$. Then $D(A)//B(L_1, L_2)$ is a bicrossed product of $L_1^{*cop}$ and $A//L_2$.*

**Proof.** Denote $D(L_1, L_2) := D(A)//B(L_1, L_2)$. Since $B(L_1, L_2) = (A//L_1)^{*cop} \bowtie L_2$ one clearly has the following inclusions of Hopf ideals $D(A)B(L_1, L_2)^+ \supset AL_2^+$ and $D(A)B(L_1, L_2)^+ \supset A^{*cop}(A//L_1)^{*cop\,+}$. These inclusions determine the following Hopf algebra embeddings:

$$A//L_2 \hookrightarrow D(L_1, L_2) \quad \text{and} \quad L_1^{*cop} := A^{*cop}//(A//L_1)^{*cop} \hookrightarrow D(L_1, L_2).$$

Moreover the multiplication map $L_1^{*cop} \bowtie A//L_2 \to D(L_1, L_2)$ given by $\overline{f} \otimes \overline{a} \mapsto \overline{(f \bowtie a)}$ is clearly surjective. A dimension argument implies that this map is bijective and therefore $D(L_1, L_2)$ is a bicrossed product of $L_1^{*cop}$ and $A//L_2$. □

## 5. Examples and applications

In this section we give some general examples of normal Hopf subalgebras of $D(A)$ together with some applications. Recall that a Hopf algebra $A$ is called simple if there are no proper normal Hopf subalgebras of $A$.

**Example 5.1.** Let $A$ be a semisimple Hopf algebra and $K(A)$ its maximal central Hopf subalgebra. Then $K(A)$ is a normal Hopf subalgebra of $D(A)$. Indeed in this case one has that $K(A) = B(A, K(A))$ and the conditions of the above theorem are satisfied.

**Example 5.2.** Let $A$ be a semisimple Hopf algebra. Then for any central Hopf subalgebra $L \subset K(A)$ it follows that $(A//L)^* \bowtie A$ is a normal Hopf subalgebra of $D(A)$. Indeed in this case one has that $(A//L)^* \bowtie A = B(L, A)$ and the conditions of the above theorem are satisfied.

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### 5.1. Drinfeld doubles of abelian extensions

**Lemma 5.3.** *Suppose that $K$ is a normal commutative Hopf subalgebra of a semisimple Hopf algebra $A$. Then $B(K, K)$ is a commutative Hopf subalgebra of $D(A)$.*

**Proof.** One has to show that $mf = f \bowtie m$ for all $m \in K$ and any $f \in (A//K)^*$. Note that using Eq. (4.1) one has $mf = (Sm_3 \rightharpoonup f \leftharpoonup m_1) \bowtie m_2 = f\epsilon(m_1)\epsilon(S(m_3)) \bowtie m_2 = f \bowtie m$. □

Recall that a Hopf algebra $A$ is called an abelian extension if there is an exact sequence of Hopf algebras

$$k \to k^G \to A \xrightarrow{\pi} kF \to k \tag{5.1}$$

for some finite groups $G$ and $F$. Next it will be shown that the Drinfeld double of an abelian extension is also an abelian extension. Note that in [20, Theorem 1.3] in order to show that $\mathrm{Rep}(D(A))$ is group-theoretical it is shown that the Drinfeld double of an abelian extension is an $R$-twist of a twisted Drinfeld double $D^{\omega}(\Sigma)$ for sum finite group $\Sigma$.

**Theorem 5.4.** *If $A$ is an abelian extension then $D(A)$ is also an abelian extension.*

**Proof.** Suppose as above that $A$ is obtained as an abelian extension:

$$k \to k^G \to A \xrightarrow{\pi} kF \to k$$

for some finite groups $G$ and $F$. Then $B(k^G, k^G)$ is a normal Hopf subalgebra of $D(A)$. Indeed note that $(A//k^G)^* \cong k^F$ is also a commutative Hopf subalgebra of $A^*$. One has the following extension

$$k \to B(k^G, k^G) \to D(A) \xrightarrow{\delta} B \to k \tag{5.2}$$

where $B := D(A)//B(k^G, k^G)$ and $\delta$ is the canonical Hopf projection. Note that $B(k^G, k^G) = (k^F)^{\mathrm{cop}} \bowtie k^G$ as a Hopf subalgebra of $D(A)$. Moreover since $B(k^G, k^G)$ is commutative by Lemma 5.3 it follows that $B(k^G, k^G) = k^{F^{\mathrm{op}} \times G}$.

On the other hand by Proposition 4.7 one has that $D(A)//B(K, K) \cong kG \bowtie kF$ is a bicrossed product Hopf algebra. Since both $kG$ and $kF$ are pointed it follows that their bicrossed product is also pointed and therefore a group algebra $kM$. □

The group $M$ from above is obtained as an exact factorization of the groups $G$ and $F$. Therefore $M = G \bowtie F$. In Theorem A.1 from the Appendix we give a complete description of the group structure of $M$. As expected, it will be shown that $M$ is isomorphic to the group $\Sigma$ from [20, Theorem 1.3].

**Theorem 5.5.** *Any normal Hopf subalgebra of $D(A)$ with trivial intersections with both $A$ and $A^*$ is a group algebra which is central in $D(A)$.*

**Proof.** Let $H$ be a Hopf subalgebra of $D(A)$ with $H \cap A = k$ and $H \cap A^* = k$. Then $L_1 = A$ and $L_2 = k$. Therefore using the notations from Theorem 3.4 it follows that $G$ is a subgroup of $G(A^*) \cap G(A)$ and

$$H = \bigoplus_{x \in G} kf_x \bowtie kg_x. \tag{5.3}$$

Since $H$ is a normal Hopf subalgebra of $D(A)$ it follows that $\Lambda_H$ is a central element of $D(A)$. Therefore $\Lambda_H = \sum_{x \in G} f_x \bowtie g_x$ is central in $D(A)$. Note that the condition $f\Lambda_H = \Lambda_H f$ implies that

$$\sum_{x \in G} ff_x \bowtie g_x = \sum_{x \in G} f_x f(g_x^{-1} ? g_x) \bowtie g_x$$

which shows that $f_x \bowtie g_x$ commutes to any element $f \in A^*$. Similarly $f_x \bowtie g_x$ commutes to any element $a \in A$ and therefore this is a central element of $D(A)$. □

**Proposition 5.6.** *Any normal Hopf subalgebra of $D(A)$ containing $A$ is of the type $(A//L)^* \bowtie A$ with $L \subseteq K(A)$. By duality any normal Hopf subalgebra of $D(A)$ containing $A^*$ is of the type $A^* \bowtie K$ with $K$ a normal Hopf subalgebra of $A$ such that $(A/K)^* \subseteq K(A^*)$.*

**Proof.** If $A \subseteq H := B(L_1, L_2, G)$ then $L_2 = A$ and therefore $G = 1$. Thus $H = (A//L_1)^* \bowtie A$ and the condition $[L_1, A] = 0$ implies that $L_1 \subseteq K(A)$. □

**Corollary 5.7.** *If $A$ is a simple noncommutative Hopf algebra then any normal Hopf subalgebra of $D(A)$ is a central group algebra.*

**Proof.** Since $A$ is a simple Hopf algebra it follows that also $A^*$ is a simple Hopf algebra. Since $K(A) = k$ it follows by previous proposition that any normal Hopf subalgebra of $D(A)$ containing $A$ equals $D(A)$. Thus any proper normal Hopf subalgebra of $D(A)$ has trivial intersections with both $A$ and $A^*$ and therefore this is a central group algebra by Theorem 5.5. □

**Proposition 5.8.** *Any minimal normal Hopf subalgebra of $D(A)$ is of one of the following types:*

(1) *$B(K, L)$ for a pair of normal Hopf subalgebras of $A$ satisfying $[K, L] = k$ and $[(A//K)^*, (A//L)^*] = k$;*
(2) *a central group subalgebra of $D(A)$.*

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**Proof.** Let $B(K, L, G)$ be a minimal normal Hopf subalgebra of $D(A)$. By Corollary 4.6 it follows that $B(K, L) \subseteq B(K, L, G)$ is also a normal Hopf subalgebra of $D(A)$. Therefore if $B(K, L) \neq B(K, L, G)$ then $B(K, L) = k$ which implies that $K = A$ and $L = k$. It follows by Eq. (3.5) that $B(K, L, G)$ has trivial intersections with $A$ and $A^*$. Thus by Theorem 5.5 it follows that $B(K, L, G)$ is a central group subalgebra of $A^*$. □

**Proposition 5.9.** *If $A \cong K \otimes L$ as Hopf algebras then $D(A) \cong B(K, L) \otimes B(L, K)$ as Hopf algebras.*

**Proof.** It is easy to see that $A^* \cong (A//K)^* \otimes (A//L)^*$ as Hopf algebras. Then clearly $B(K, L)$ and $B(L, K)$ are normal Hopf subalgebras of $D(A)$. Moreover $B(K, L) \cap B(L, K) = k$ and $B(K, L)B(L, K) = D(A)$. It follows by [6, Theorem 2.5] that $D(A) \cong B(K, L) \otimes B(L, K)$ as Hopf algebras. □

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## Appendix. Drinfeld doubles of abelian extensions

### A.1. Abelian extensions

Let $\Sigma = FG$ be an an exact factorization of finite groups. This gives a right action $\triangleleft : G \times F \to G$ of $F$ on the set $G$, and a left action $\triangleright : G \times F \to F$ of $G$ on the set $F$ subject to the following two conditions:

$$s \triangleright xy = (s \triangleright x)((s \triangleleft x) \triangleright y) \qquad st \triangleleft x = (s \triangleleft (t \triangleright x))(t \triangleleft x). \tag{A.1}$$

The actions $\triangleright$ and $\triangleleft$ are determined inside the group $\Sigma$ by the following relation

$$gx = (g \triangleright x)(g \triangleleft x) \tag{A.2}$$

for all $x \in F, g \in G$. Note that $1 \triangleright x = x$ and $s \triangleleft 1 = s$. The quadruple $(G, F, \triangleleft, \triangleright)$ is also called a matched pair of groups.

Consider the Hopf algebra $A = k^G\ {}^{\tau}\#_{\sigma}\ kF$ from [17] which is a crossed product and coproduct formed using the above two actions. Recall that the structure of $A$ is given by:

$$(p_g\#x)(p_h\#y) = \delta_{g\triangleleft x,h}\sigma_g(x, y)p_g\#xy \tag{A.3}$$

$$\Delta(p_g\#x) = \sum_{st=g} p_s\#(t \triangleright x) \otimes \tau_x(s, t)p_t\#x \tag{A.4}$$

where $p_g \in k^G$ is is the dual basis to the group element basis of $kG$. Without loss of generality we may suppose that $\tau_x(g, g^{-1}) = 1 = \sigma_g(x, x^{-1})$ for all $g \in G$ and $x \in F$ (see [11, Lemma 3.6]). Moreover in this case the antipode of $A$ is given by

$$S(p_g\#x) = (g \triangleright x)^{-1}p_{g^{-1}} = p_{(g\triangleleft x)^{-1}}\#(g \triangleright x)^{-1}. \tag{A.5}$$

Recall that here $\sigma : F \times F \to k^G$ is a normalized two cocycle written as $\sigma(x, y) := \sum_{g\in G} \sigma_g(x, y)p_g$ Similarly, $\tau : G \times G \to k^F$ is a normalized two cocycle written as $\tau(g, h) := \tau_x(g, h)q_x$ where $\{q_x\}_x \subset k^F$ is the dual basis to the group element basis of $kF$.

Then the Hopf algebra $A$ from above fits into the abelian extension

$$k \longrightarrow k^G \xrightarrow{i} A \xrightarrow{\pi} kF \longrightarrow k. \tag{A.6}$$

Note that under the identification $A = k^G\ {}^{\tau}\#_{\sigma} kF$ one has that $\pi(p_g\#x) = \delta_{g,1}x$ and $i(p_g) = p_g\#1$. This induces an isomorphism $A//k^G \to kF$ via $\overline{p_a \otimes x} \mapsto \delta_{a,1}x$.

Moreover, as it is shown in [17] any abelian Hopf algebra fitting an exact sequence as in Eq. (A.6) is of the form presented above.

### A.2. Dual Hopf algebra of an abelian extension

The dual Hopf algebra $A^*$ fits the exact sequence

$$k \longrightarrow k^F \xrightarrow{\pi^*} A \xrightarrow{i^*} kG \longrightarrow k \tag{A.7}$$

which shows that this is also an abelian extension.

Under the obvious identification $A^* \cong k^F\ {}^{\sigma^*}\#_{\tau^*}\ kG$ it can be shown that

$$(q_x \otimes a)(q_y \otimes b) = \delta_{x,b\triangleright y}\tau_y(a, b)q_y\#ab \tag{A.8}$$

$$\Delta(q_x \otimes a) = \sum_{uv=x} (q_u \otimes a) \otimes (\sigma_a(u, v)q_v \otimes (a \triangleleft u)). \tag{A.9}$$

Note that in this case $A^*//k^F \xrightarrow{\cong} kG$ via $\overline{q_x \otimes a} \mapsto \delta_{x,1}a$.

### A.3. On the Drinfeld double of abelian extensions

In this subsection we describe the group $M = G \bowtie F$ from the proof of Theorem 5.4. More precisely we prove the following.

**Theorem A.1.** *Suppose that $A$ is an abelian extension fitting the exact sequence of Eq. (A.6). Then $D(A)$ fits the following exact sequence*

$$k \to k^{F^{\mathrm{op}} \times G} \to D(A) \xrightarrow{\delta} k(G \bowtie F) \to k \tag{A.10}$$

*where the factorization of $G \bowtie F$ is given via the actions $x \blacktriangleright a = (a^{-1} \triangleleft x^{-1})^{-1}$ and $x \blacktriangleleft a = (a^{-1} \triangleright x^{-1})^{-1}$ for all $a \in G$ and $x \in F$.*

In the proof below, for shortness of writing, we denote the linear functional $b \rightharpoonup f \leftharpoonup a \in A^*$ by $f(a?b)$ for any $f \in A^*$ and any $a, b \in A$. Note that $f(a?b)(x) = f(axb)$ for all $x \in A$.

**Proof.** Since $G^{\mathrm{op}} \bowtie F$ is an exact factorization of groups it follows that $xa = (x \blacktriangleright a)(a \blacktriangleleft x)$ for some matched pair of groups on $G^{\mathrm{op}}$ and $F$.

One has that $D(A) \cong A^* \bowtie A$ as vector spaces and the canonical projection

$$D(A) \xrightarrow{\delta} k(G^{\mathrm{op}} \bowtie F) \tag{A.11}$$

from Theorem 5.4 is given by $\overline{q_x \otimes b} \bowtie \overline{p_a \# y} \mapsto \delta_{x,1}\delta_{a,1} b \bowtie y$. Note that inside the quotient $k(G^{\mathrm{op}} \bowtie F)$ one has $xa = \delta((\overline{p_1 \# x})(\overline{q_1 \otimes a}))$. On the other hand inside $D(A)$ it follows

$$(p_1\#x)(q_1 \otimes a) = \langle q_1 \otimes a, S(p_1\#x)_3?(p_1\#x)_1\rangle (p_1\#x)_2.$$

Note that since we assumed $\tau_x(s, s^{-1}) = 1$ by Eq. (A.4) one has

$$\begin{aligned}\Delta^2(p_1\#x) &= \sum_{s\in G}(p_s\#(s^{-1} \triangleright x)) \otimes \Delta(p_{s^{-1}}\#x)\\ &= \sum_{m,n\in G}(p_{(mn)^{-1}}\#(mn \triangleright x) \otimes (p_m\#(n \triangleright x)) \otimes \tau_x(m, n)(p_n\#x).\end{aligned}$$

This implies that

$$\begin{aligned}(p_1\#x)(q_1 \otimes a) &= \langle q_1 \otimes a, S(p_1\#x)_3?(p_1\#x)_1\rangle (p_1\#x)_2\\ &= \sum_{m,n\in G}\langle q_1 \otimes a, S(p_n\#x)\tau_x(m, n)?(p_{(mn)^{-1}}\#(mn \triangleright x))\rangle \bowtie\\ &\quad \bowtie (p_m\#(n \triangleright x)).\end{aligned}$$

It follows that

$$\begin{aligned}\delta((p_1\#x)(q_1 \otimes a)) &= \sum_{m,n\in G}\delta(\langle q_1 \otimes a, S(p_n\#x)?(p_{(mn)^{-1}}\#(mn \triangleright x))\rangle) \times \delta((p_m\#(n \triangleright x)))\\ &= \sum_{n\in G}\delta(\langle q_1 \otimes a,\ S(p_n\#x)?(p_{n^{-1}}\#(n \triangleright x))\rangle)(n \triangleright x).\end{aligned}$$

Denote $f^n_{x,a} := \langle q_1 \otimes a,\ S(p_n\#x)?(p_{(mn)^{-1}}\#(mn \triangleright x))\rangle \in A^*$. Therefore the above formula can be written as

$$xa = \delta((p_1\#x)(q_1 \otimes a)) = \sum_{n\in G}\delta(f^n_{x,a})(n \triangleright x). \tag{A.12}$$

Note that for any $f \in A^*$ one has

$$f = \sum_{g\in G,\ x\in F} f(p_b\#x)(q_x \otimes b). \tag{A.13}$$

Therefore $\pi(f) = \sum_{b\in G} f(p_b\#1)b$. For $f^n_{x,a}$ it follows that

$$\begin{aligned}\pi(f^n_{x,a}) &= \sum_{b\in G} f^n_{x,a}(p_b\#1)b\\ &= \langle q_1 \otimes a,\ (S(p_n\#x))(p_b\#1)(p_{n^{-1}}\#(n \triangleright x))\rangle b\\ &= \langle q_1 \otimes a,\ (n \triangleright x)^{-1}p_{n^{-1}}\#(n \triangleright x)\rangle n^{-1}\\ &= \delta_{b,n^{-1}}\langle q_1 \otimes a,\ p_{n^{-1}\triangleleft(n\triangleright x)}\#1\rangle n^{-1}\\ &= \langle q_1 \otimes a,\ p_{(n\triangleleft x)^{-1}}\#1\rangle n^{-1}\\ &= \delta_{a,(n\triangleleft x)^{-1}}n^{-1} = \delta_{a^{-1},(n\triangleleft x)}n^{-1} = \delta_{a^{-1}\triangleleft x^{-1},n}(a^{-1} \triangleleft x^{-1})^{-1}.\end{aligned}$$

Thus Eq. (A.12) implies

$$xa = \sum_{n \in G} \delta(f^n_{x,a})(n \triangleright x) = (a^{-1} \triangleleft x^{-1})^{-1}((a^{-1} \triangleleft x^{-1}) \triangleright x). \tag{A.14}$$

This implies that $x \blacktriangleright a = (a^{-1} \triangleleft x^{-1})^{-1}$ and $x \blacktriangleleft a = (a^{-1} \triangleleft x^{-1}) \triangleright x$. It follows from Eq. (A.1) that $(a^{-1} \triangleleft x^{-1}) \triangleright x = (a^{-1} \triangleright x^{-1})^{-1}$. Thus $x \blacktriangleleft a = (a^{-1} \triangleright x^{-1})^{-1}$. □

**Remark A.2.** It follows that $M \cong \Sigma$ by the map $(a, x) \mapsto (x, a)$ since from Eq. (A.2) the multiplication inside $\Sigma$ verifies $xa = (a^{-1}x^{-1})^{-1} = (a^{-1} \triangleright x^{-1})^{-1}(a^{-1} \triangleleft x^{-1})^{-1}$ for all $a \in G$ and all $x \in F$.

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